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On the ground states of the frustration model of a spin glass by a matching method of graph theory

I Bieche†, R Maynard‡, R Rammal‡ and J P Uhry†

† Recherche Opérationnelle, Laboratoire IMAG associé au CNRS, Université Scientifique et Médicale de Grenoble 1, BP 53 X, 38041 Grenoble Cedex, France

‡ Centre de Recherches sur les Très Basses Températures, CNRS, BP 166 X, 38042 Grenoble Cedex, France

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Abstract. The ground states of a quenched random Ising spin system with variable concentration of mixed nearest-neighbour exchange couplings $\pm J$ on a square lattice (frustration model) are studied by a new method of graph theory. The search for ground states is mapped into the problem of perfect matching of minimum weight in the graph of frustrated plaquettes, a problem which can be solved by the algorithm of Edmonds. A pedestrian presentation of this elaborated algorithm is given with a discussion of the condition of validity.

The simulation of finite square lattices of variable size from 10×10 up to 22×22 and various concentrations x ranging from 0·1 to 0·3 is performed. The main results are a new determination of the threshold of disappearance of ferromagnetism $x^*=0{\cdot}145$, the probability of occurrence of a fracture line through the samples, the fraction of spins inside the connected components measuring the looseness of ground states and the average magnetisation as a function of x.

In contrast to the standard relaxation methods often used in this context, it is emphasised that this method is the only one which can generate the exact ground states of the frustration model in a reasonable amount of computing time.

1. Introduction

The problem of spin glasses is of growing interest both in solid state physics and in statistical physics. This problem first emerged from the observed properties of dilute magnetic alloys, such as 1% of Mn or Fe embedded in Cu or Au. The most striking property of these systems is the cusp in the magnetic susceptibility at a well defined temperature (at least for the low-frequency measurement) but an absence of a peak in the specific heat and only a broad maximum at another temperature. From this behaviour, the question of a new phase transition at low temperature arose; this would be distinct both from the disordered high-temperature paramagnetic phase and from the well ordered ferromagnetic phase.

In these alloys, two spins of the magnetic atoms S_i and S_j are coupled via the Ruderman–Kittel–Kasuya–Yosida interaction $J(r_{ij})S_iS_j$, in which $J(r_{ij})$ varies as $\cos(2k_Fr_{ij})/r_{ij}^3$ where r_{ij} is the distance between the impurities and k_F is the Fermi wavevector of the conduction electrons ($k_Fr_{ij}\gg1$).

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These oscillations show that the sign of J can be positive or negative depending on r_{ij}. When the magnetic impurities are randomly distributed in the lattice of the metal, one expects a great variety of different interactions between spins. The distribution of spins, $p(J)$, is centred at $J = 0$ and is symmetric in J.

The first step in modelling this problem is to substitute for the alloy a regular lattice of spins with random interaction. Each bond of the lattice of spins corresponds to a value of J which is chosen at random from the probability distribution $p(J)$. But when the lattice is built, the value of each bond is fixed and there is no possibility of changing it: this corresponds precisely to the quenched problem of spin glasses.

The second step towards a simple model is to use Ising spins S_i for which values can be only ± 1. Only a very simple lattice, the square lattice in two dimensions, will be considered here. Finally, it is assumed that the interaction can have only two symmetrical values $\pm J$, the density of $-J$ bonds being x and that for $+J$ being $1-x$. Then the distribution function has only two values $p(J') = x\delta(J'+J) + (1-x)\delta(J'-J)$. This is the $\pm J$ model, or frustration model, which will be studied in this paper. It is believed that despite its strong simplifications, this model retains the relevant features of real spin glasses.

This model, as well as models with a continuous distribution of J, has been studied analytically by various approximate methods. The first attempt to calculate a partition function in the framework of the mean-field approximation was performed by Edwards and Anderson (1975), who concluded that there is a non-zero order parameter below a critical temperature. This order parameter, contrasting with the standard theory of phase transitions, is purely local and describes the freezing of the spins in their own local direction below T_g. This theory raised for the first time the problem of a phase transition in a spin glass system and prompted a large response from theoreticians. But the approximation used—the replica method—has been controversial and, despite various other methods of approximation used in this context, there is no firm evidence today of the occurrence of a phase transition, except perhaps in a special model of infinite range of interaction (Sherrington and Kirkpatrick 1975).

As the situation is so open, with no results firmly established, we turned towards this very simplified model in the hope that any progress towards comprehension here might help in the understanding of more realistic situations.

This article is devoted to the study of ground states of the frustration model (i.e. at $T = 0$, but when x varies from $x = 0$ (ferromagnetic ground state) to $x = 0{\cdot}5$ (the spin glass ground state), the phase diagram being symmetric around $x = 0{\cdot}5$). Particular attention will be paid to the threshold x^* above which the ferromagnetism disappears. The determination of this value, together with the characterisation of ground states in terms of clusters of solidary spins (clusters of spins which can be reversed simultaneously without any cost in energy) and of defect lines or fractures (which run from one side to the other of a finite sample permitting the spins of a large portion of the sample to be reversed), are the objectives of this work.

These objectives have been tentatively pursued recently by the well known Monte-Carlo relaxation method (see, e.g., the review article by Binder (1978)). A Monte-Carlo step consists of choosing a spin at random, calculating the energy change ΔE associated with flipping this spin and flipping the spin with probability 1 if $\Delta E < 0$ or with probability $\exp(-E/k_B T)$ if $\Delta E \geqslant 0$. Recently, however, it has been realised (see, e.g., Bray and Moore 1977, Stauffer and Binder 1978) that the Monte-Carlo relaxation can give neither the thermodynamic equilibrium states nor the ground states for the spin glass model. The difficulties of these relaxation models dwell in the high potential

barrier and the high degeneracy which appears in some cases at low temperatures, when the spins are requested to flip (Rammal *et al* 1979). The algorithm used here is not relaxational and overcomes such difficulties; it permits the generation of the ground states in a relatively short computational time. It is therefore very well suited to this problem and is more efficient than all other known methods.

The starting point of the method is to be found in the paper by Toulouse (1977). It is shown that the property of local gauge invariance leads to the definition of frustrated plaquettes in the $\pm J$ Ising model as a square of spins where the number of negative bonds is odd on the perimeter. It is noticed that each frustrated plaquette is a source or sink for a set of unsatisfied bonds called elongation. 'A ground-state configuration is obtained by associating the frustrated plaquettes in pairs, in such a way that the sum of the elongation of the pairs is minimal'. This procedure has been carried out by hand in further papers (Vanimenus and Toulouse 1977, Vanimenus *et al* 1979) on samples of finite size and a determination of the threshold and ground-state energy has been proposed.

The present investigation is purely computational: the accuracy of numerical results has been improved, particularly in the definition of the threshold, and some important remarks on the limitation of the method have been put forward.

Since the method of graph matching does not seem to have been used previously in statistical physics, we start with a description of the algorithm in an intuitive, rather than rigorous, way. It is hoped that this procedure could have applications in other contexts in statistical physics. The reader who wants to acquire greater expertise with this algorithm is advised to refer to a more rigorous version (Bieche 1979).

In § 2 the problem of finding a ground state for the spins on a square lattice is mapped progressively into the problem of perfect matching of minimum weight in the graph of frustrated plaquettes, where, as quoted previously, the weight of an edge is precisely defined as the chain of minimum length (elongation) between two frustrated plaquettes. Careful consideration is given to the conditions for validity of this mapping. The periodic boundary conditions so often used in this context do not fall in the domain of validity here, nor does the three-dimensional problem (on a simple cubic lattice, for instance).

Section 3 presents the algorithm of Edmonds (1965a, b) used for generating the perfect matching of minimum weight. Firstly, in order to use results of linear programming, a first transposition of the basic problem is performed; this shows the important role played by the odd subset of vertices. These odd subsets imply additional constraints, as shown by Edmonds, the physical meaning of which is not clear for the moment; however, they are essential to the strategy of the algorithm. At this stage the most efficient method of solving the problem consists of building the dual problem of the matching, by defining one pair of dual vertex variables u_i and u_j for each edge (i, j). Next a relation is obtained between the u_i's and the special variables of the odd subsets; this permits one to generate the perfect matching of minimum weight (ground state). The efficiency of this algorithm comes from the fact that the 'variational problem' is mapped at each step in the problem of finding a matching of maximum cardinality. Although the initial state is chosen by a criterion of proximity, this algorithm does not progress by local minimisation and reveals itself to be well suited to handling the highly *nonlocal* character of the frustration model. In contrast to the standard relaxation method (such as the Monte-Carlo calculations) there is no transient regime where the matching would progress towards the optimal solution: as the matching becomes perfect at the end of calculation, it is optimum at the same time. To acquire a certain

familiarity with this method, which is more sophisticated than the standard method used in the context of statistical physics, the strategy of the algorithm is illustrated for a simple example in the appendix.

Section 4 is devoted to a characterisation of the spin configuration of the ground states. The algorithm of Edmonds permits the generation of one of the ground states, whose total number cannot be determined at the present state of affairs. However, it is relatively easy to exhibit the connected components of the graph, i.e. the subset of the plaquettes which are matched together in *any* perfect matchings of minimum weight (ground state). Moreover, it is easy to establish that two distinct perfect matchings M and M' are possible such that the symmetrical difference $M\Delta M'$ generates some alternating cycles (along the cycle the edges belong alternately to M or M') of zero weight. On the lattice of spins these alternating cycles surround clusters of spins which can be reversed by going from M to M' without any cost in energy. The existence of these clusters of spin, as well as alternating cycles, has already been noticed (Bray *et al* 1978), and have been called clusters of solidary spins and the contour of zero energy, respectively. The interesting property of these clusters is that they belong to a connected component of the graph. This algorithm is therefore capable of producing the fraction of spins which belong to a connected component, a quantity which appears to be critical and relevant for characterising the looseness of the ground states.

The other very critical property is the occurrence in samples of finite size of a long alternating cycle running from one side of the sample to the opposite one. This fracture line, which is the signature of the vanishing of ferromagnetism, must be found in the largest connected component spreading from one boundary to the opposite one (the 'percolating' component). Such fracture lines were found by hand, using a knowledge of the connected components produced by the algorithm. This restricted the investigation to a small part of the sample.

The results of the simulation are reported in § 5 for (i) the energy of the ground state, E_0, as a function of x for square lattices up to size 22×22, (ii) the probability of occurrence of fracture lines as a function of x, (iii) the fraction of spins inside the connected components and (iv) the average magnetisation.

The last three properties permit a determination of the critical concentration for the disappearance of ferromagnetism $x^*\simeq 0{\cdot}145\pm0{\cdot}01$. It is striking that this threshold value coincides with the one obtained using the annealed model (in this model the bonds J_{ij} are considered as internal variables, like the spins), $x^*=0{\cdot}1464$, for which an exact solution exists.

These results have been obtained on a square lattice but can be generalised without difficulties to any two-dimensional lattice.

2. Frustration and matching problems

2.1. Definitions

In this section we show how the problem of finding the ground states in a frustrated network can be reduced to a perfect matching of minimum weight in the frustration graph.

The most important concepts from graph theory for our purpose are those which deal with matching notions. Before attempting the study of matching properties of frustrated networks, it is useful to recall some important concepts as well as a consistent set of definitions and notational conventions (see Berge 1962).

A graph $G = (V, E)$ is a structure consisting of a finite set V of elements called vertices, and a set E of unordered pairs of vertices called edges. The degree, denoted $d_G(v)$, of a vertex v is the number of edges incident to v. A cycle c in $G = (V, E)$ is a sequence of edges, of the form $(v_1, v_2), (v_2, v_3), \ldots, (v_p, v_1)$ where p is a positive integer. A subset $M \subset E$ is said to be a *matching* if, in the graph $G_M = (V, M)$, the degree of each vertex is less than or equal to one. A vertex v is *saturated* by M if and only if $d_{G_M}(v) = 1$ (i.e. if an edge of M incident to v exists). If $d_{G_M}(v) = 0$, v is said to be unsaturated. A matching M is perfect if and only if all vertices of G are saturated (i.e. M saturates every vertex in G).

Given a graph $G = (V, E)$ and a mapping $\omega : E \to R$, called the *weight* function, for any subset $M \subset E$, we define the weight of M as

$$\omega(M) = \sum_{e \in M} \omega(e).$$

In such a case, G is called *a weighted graph.*

With a matching in a weighted graph, we associate a new weight function ω^M, relative to M, by

$$\omega^M(e) = \begin{cases} \omega(e) & \text{if } e \in M \\ -\omega(e) & \text{if } e \notin M. \end{cases}$$

If M denotes a matching in $G = (V, E)$, an *alternating* cycle relative to M is a simple cycle whose edges are alternately in M and $\bar{M} = E - M$.

In the following, we are interested in graphs of the form $G = (S, E)$, where S denotes the locations of the spins in a lattice and E is a representation of interactions (coupling) between spins. Associated with $G = (S, E)$ is the weight function γ, where $\gamma(e)$ represents the value of the coupling between spins i and j of $e = (i, j)$.

2.2. *Frustration and matching*

Let us consider a graph $G = (S, E)$ where each vertex i is assigned a spin $S_i = \pm 1$. With each pair of spins S_i and S_j, which are the ends of the edge $e = (i, j) \in E$, we associate an interaction energy of the form $-J_{ij}S_iS_j$, where J_{ij} denotes a given real number, representing the intensity of interaction. For example, in the frustration problem J_{ij} can be positive (ferromagnetic coupling) as well as negative (antiferromagnetic coupling).

It is not hard to see that, for every cycle c of G containing an *odd* number of negative assigned edges, there exists no configuration of spins in c such that every edge has its minimum energy. In this case, the cycle c is said to be *frustrated.* An edge having its minimum energy is to be called *satisfied.* Thus, there is no difficulty in deducing the following statement.

Theorem 1. It is possible to associate a configuration of spins with a given set of unsatisfied edges L if and only if every frustrated (unfrustrated) cycle in the graph has an odd (even) number of edges in L. We call this condition (C).

In the following, we call a set of unsatisfied edges such that the condition (C) is fulfilled a *solution* (or set of frustrations).

This weak theorem can be replaced by a stronger one. In fact, we can show (Bieche 1979) that the condition (C) can be limited to a subset P of the family $\mathscr{C}$ of cycles in G with the following property: for every $C \in \mathscr{C}$, there exists a subset $\{p_1, \ldots, p_k\} \subset P$ such

that $C = p_1 \Delta p_2 \Delta \ldots \Delta p_k$. Here Δ is the symmetrical difference operation. The elements of P are called *plaquettes*.

Next we discuss two examples.

(i) On a simple square network, belonging to an infinite planar square lattice, it is easy to see that the elementary square cycles with four edges are a P family of plaquettes (figure 1).

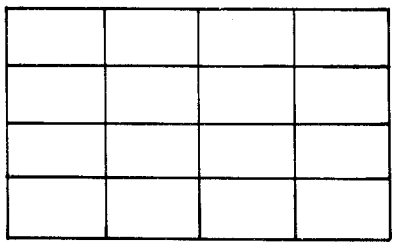

Figure 1. Plaquettes in a planar square network.

(ii) On a toroidal network the two cycles illustrated in figure 2 cannot be obtained from a symmetrical difference between elementary square cycles and have to be added to the P family. These new cycles introduce a supplementary difficulty which originates in the modification of the genus associated with the given graph.

Therefore, we limit our reasoning to a *planar* finite square grid G (planar graph). The results established in the following can be extended without difficulty to any planar graph.

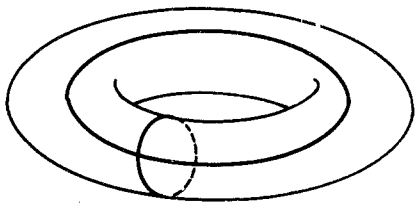

Figure 2. Toroidal network corresponds to standard periodic boundary conditions: the two new cycles which cannot be obtained by symmetrical difference of elementary square cycles.

The first important consequence of the planarity of the graph considered is the existence of a P family such that each edge belongs to exactly two plaquettes. This naive property plays an important role in our investigation.

We associate with the given network the graph $G' = (P, E')$, where P is the set of plaquettes (i.e. vertices of G') and E' (edges of G') is the set of pairs (p_1, p_2) where p_1 and p_2 denote two plaquettes having a common edge on the network. G' is, in fact, a special representation of the dual graph associated with the network G. In particular, there is a bijection between the edges of G' and those of G (figure 3).

For a given realisation of the interactions J_{ij} on G, we note that F is the set of vertices of G' corresponding to frustrated plaquettes. If we denote the number of elements in F by $|F|$, it is easy to see that $|F|$ can be assumed to be even. In fact, in all cases we can add an appropriate frustrated cycle, without any modification to the problem, and obtain in

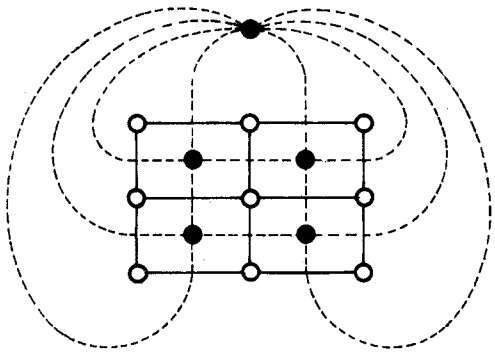

Figure 3. Correspondence between the edges of G' and those of G: G corresponds to {○, —} while G' corresponds to {●, - - - -}.

this way an even value for $|F|$. With these prescriptions, each edge (p_1, p_2) in G' has an assigned weight $\delta(p_1, p_2)$ which is the amplitude of the interaction represented by the common edge to p_1 and p_2 in G.

Given a spin configuration $\mathscr{L}$ in G, we define the graph $G'(\mathscr{L}) \equiv (P, \mathscr{L}')$ where $\mathscr{L}'$ is the set of edges in G' associated with the solution $\mathscr{L}$. $\mathscr{L}'$ is simply the set of dual edges of the elements in $\mathscr{L}$.

To characterise a solution $\mathscr{L}$, we can show the following result. A set of edges $\mathscr{L}$ in G yields a solution if and only if $d_{G'(\mathscr{L})}(f)$ is odd for every $f \in F$ and $d_{G'(\mathscr{L})}(p)$ is even for every $p \in P - F$. The weight of this solution $\mathscr{L}$ is by definition

$$\delta(\mathscr{L}) = \sum_{e \in \mathscr{L}} \delta(e).$$

This statement means simply that every configuration of spins, minimising the total energy, corresponds to a solution of minimum weight. In such a solution some conditions are imposed on the degrees of the elements in G'. A simple consequence is that in $G'(\mathscr{L})$, where $\mathscr{L}$ is a solution, there is no cycle. As a matter of fact, if one cycle is present, the subtraction of this cycle from $\mathscr{L}$ leads to a better solution satisfying both the previous conditions.

From the above statement, we deduce a more precise characterisation of the solution $\mathscr{L}$.

Theorem 2 (Bieche 1979). Given a solution $\mathscr{L}$ of minimum weight, there exists a partition of edges in $G'(\mathscr{L})$ into $\frac{1}{2}|F|$ chains, each connecting two vertices belonging to F and having no common edge.

Corollary. Any solution of minimum weight is given by a perfect matching of minimum weight in the graph $K = (F, F \times F)$. The weight $\omega(f_1, f_2)$ of the edge (f_1, f_2) is simply the weight of the minimum-weighted chain connecting f_1 with f_2 in G'.

Proof. Following theorem 2, each solution of minimum weight gives rise to a perfect matching in K, where the weight is nothing other than the total sum of minimum-weighted chains. On the other hand, each minimum-weight perfect matching in K yields a set of chains and then a solution of the same weight.

Note here that a perfect matching is possible in all cases because of the assumption of $|F|$ being even. A minimum-weighted perfect matching can give more than one solution of minimum weight. This is the case when at least one edge (f_1, f_2) exists in the matching with more than one minimum-weighted chain in G'.

To conclude this section, we recall the main result established above. The determination of ground states in a frustrated planar network is equivalent to the problem of finding a minimum-weighted perfect matching in an appropriate graph. The definition of the weight function is given by the interactions J_{ij} assigned to the edges of the network. This conclusion is the starting point of the next section, where we show how to find a minimum-weighted perfect matching.

3. A minimum-weighted perfect matching by Edmonds' algorithm

The theory and algorithmic techniques of matching problems have been studied in detail by specialists of graph theory. Historically, this problem was first solved for bipartite graphs, and only recently for non-bipartite graphs. The case of bipartite graphs provides some conceptual simplification, due to the absence of odd cycles, and 'blossoms' as shown by Edmonds (1965a,b). The presence of 'blossoms' involved in finding the matching provides a great complication, and a proper treatment is needed in this case.

In this section, we do not attempt a detailed study of this problem. Readers unfamiliar with notions exposed here can consult appropriate references (Lawler 1976, Nemhauser and Garfinkel 1972). Our purpose is only to illustrate essential ingredients in Edmonds' algorithm. For this purpose, we show how this algorithm works by using some illustrative examples at each stage.

Without loss of generality, we limit our investigations to the case $J_{ij} = \pm 1$. The general case with another distribution J'_{ij} can be dealt with according to the same procedure.

The graph considered here has a set of vertices, the frustrated plaquettes. We denote this set by V. Edges are the elements of $V \times V \equiv E$. Given the assumption $J_{ij} = \pm 1$, the weights associated with edges in $V \times V$ are simply the distances between plaquettes (the Manhattan metric).

With each edge $e = (i, j)$ in this graph, we associate a real number x_{ij}, having the value one if e belongs to the matching and zero otherwise.

Let W_{ij} be the weight assigned to the edge $e = (i, j)$. The problem (P1) of the search for minimum-weighted perfect matching can be formulated as follows.

$$
\begin{array}{lll}
\text{(P1)} & x_{ij} \in \{0, 1\} & \text{for every edge } (ij) \\
 & \sum_j x_{ij} = 1 & \text{for every vertex } i \in V \\
 & \sum_{(ij)} x_{ij} W_{ij} \equiv W(\{x\}) & \text{to be minimised.}
\end{array}
$$

(P1) is then trivially an integer linear programming problem with variable x_{ij} and $|V|$ constraints.

In this form, the condition $x_{ij} \in \{0, 1\}$ for every edge (ij) makes this problem (P1) intractable. If we relax this condition by imposing only $x_{ij} \geqslant 0$, we obtain the following problem (P2).

(P2) $x_{ij} \geqslant 0$ for every edge (ij)

$\sum_j x_{ij} = 1$ for every vertex $i \in V$

$\sum_{(ij)} x_{ij} W_{ij} \equiv W(\{x\})$ to be minimised.

This simplification introduces in counterpart a complication due to the fact that in the optimal solution of (P2), x_{ij} can be non-integer. In fact, we can obtain $x_{ij} = 0$, 1 or $\frac{1}{2}$. In this case, edges affected with weight $\frac{1}{2}$ are necessarily disjoint odd cycles. Moreover, in the general case, optimal solutions of (P2) are not optimal solutions for (P1) (see Nemhauser and Garfinkel 1972, p 85). In figure 4 we show an example illustrating such situations. Therefore the relaxation of the restriction $x_{ij} \in \{0, 1\}$ implies the necessity to consider with caution the family of odd subsets S of V.

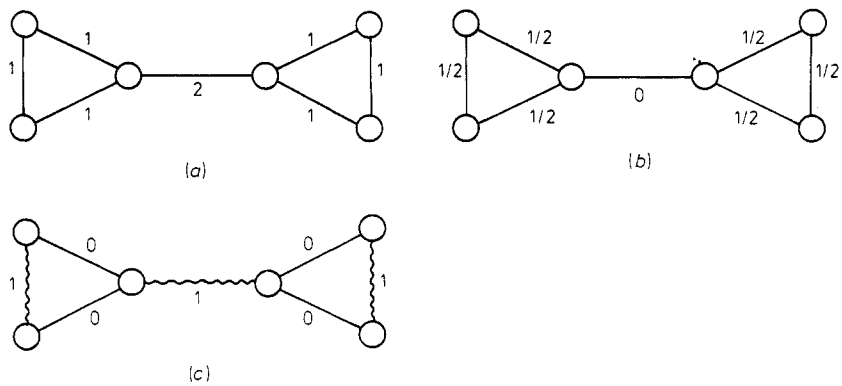

Figure 4. Example showing the importance of odd sets. (*a*) Graph with weighted edges; (*b*) solution of problem (P2), with non-integer x_{ij}, $W = 3$; (*c*) the exact solution for (P1), with $W = 4$.

An elegant solution to this problem has been derived by Edmonds taking into account the odd subsets of V. Briefly, Edmonds' approach consists of adding a new constraint associated with S. More precisely, Edmonds shows that the problem of finding a minimum-weighted perfect matching (i.e. problem (P1)) is equivalent to the following linear programming problem (P3).

(P3) $x_{ij} \geqslant 0$ for every edge (ij) (C1)

$\sum_j x_{ij} = 1$ for every vertex i (C2)

$\sum_{(ij) \in E(S)} x_{ij} \leqslant \frac{1}{2}(|S| - 1)$ for every odd subset S of V (C3)

$\sum_{(ij)} x_{ij} W_{ij} \equiv W(\{x\})$ to be minimised. (C4)

Here $E(S)$ denotes the set of edges having its end in S, i.e. $E(S) = \{(ij) \in E \mid i \in S, j \in S\}$ and $|S|$ is the cardinality of S.

From Edmonds' papers (1965a, b), the problem (P3) yields an integer basic solution $x_{ij} = 0$ or 1 for any given set of edge weights $\{W_j\}$. Note here that the complications due

to S cannot exist in bipartite graphs (i.e. graphs without odd cycles). In the frustration problem, the physical significance of this aspect is not clear.

The transposition of (P1) into (P3) is the key in the construction of minimum-weighted perfect matching.

3.1. Linear programming formulation of the minimum-weighted matching problem

To our knowledge, the most efficient procedure for solving problem (P3) is based on the duality theory in linear programming (Dantzig 1962). The basic idea of duality is that every linear programming problem is associated with another problem, called its dual, and that both problems bear such a close relationship that whenever one is solved, the other problem is in fact solved as well. In this context, (P3) is the primal problem.

The dual of the linear programming problem (P3) above is (D3).

(D3) $Z_S < 0$ for every odd subset of V (C5)

$$u_i + u_j + \sum_{\{S|(ij)\in E(S)\}} Z_S < W_{ij} \qquad \text{for every edge } (ij) \in E \qquad \text{(C6)}$$

(u_i is a real number for every vertex $i \in N$)

$$\sum_{i\in V} u_i + \sum_{\{S\}} \tfrac{1}{2}(|S|-1)Z_S \equiv T(\{u\},\{Z\}) \qquad \text{is to be minimised.} \qquad \text{(C7)}$$

Here u_i and Z_S are identified with vertex i and the odd subset S.

Using the weak theorem of complementary slackness (Dantzig 1962), we can derive the following criterion for the optimality of primal and dual solutions.

Optimality criterion. If M is a *perfect* matching such that for every edge $(ij) \in M$

$$u_i + u_j + \sum_{\{S|(ij)\in E(S)\}} Z_S = W_{ij} \qquad \text{(C8)}$$

and for every S such that $Z_S < 0$, we have

$$\sum_{(ij)\in E(S)} x_{ij} = \tfrac{1}{2}(|S|-1) \qquad \text{(C9)}$$

then M has the minimum weight. Note that the inverse of this theorem is true. In other terms, M yields a minimum-weighted perfect matching if and only if there exist two mappings U and Z such that conditions (C5), (C6), (C8) and (C9) are satisfied. This remark is very useful, as shown in the next section.

With this optimality criterion, we are now in a position to give the principle of the algorithm used through the next sections. In the appendix, we show with two examples how this algorithm works, illustrating the difficulties encountered during the computations.

3.2. Summary of the minimum-weighted perfect matching algorithm

(0) Start with a set of values u_i and Z_S according to conditions (C5) and (C6). By example, we can choose initially $u_i = \frac{1}{2}\min_j W_{ij}$ for every vertex i and $Z_S = 0$ for every S.

(1) We retain in G all edges satisfying conditions (C8) and (C9). In this way, we obtain a new graph denoted by $\bar{G}$. In the sequel we call $\bar{G}$ *the equality graph.*

(2) In $\bar{G}$, we find a matching of maximum cardinality such that the condition (C9) is satisfied.

(3) If the matching obtained is perfect, then this matching and the dual solution are optimal and we can stop. Otherwise, we modify the variables $\{u\}$ and $\{Z\}$ to increase $\bar{G}$ (i.e. add new edges), without any violation of conditions (C5), (C6), (C8) and (C9). We then return to step (2).

The implementation of the algorithm outlined above is illustrated with more details in the appendix. To end this section, it is interesting to make some remarks concerning this algorithm.

(*a*) Firstly, the complexity of this algorithm is polynomial: the original algorithm of Edmonds works in $O(|V|^4)$ operations and the best known works in $O(|V|^3)$ (Lawler 1976).

(*b*) In contrast to techniques used previously in the frustration problem, the algorithm used here is not based on a relaxation method. Moreover, there is no transient solution. The algorithm leads up to an optimal solution, i.e. a ground state in a frustrated network.

(*c*) The computational procedure maintains primal and dual aspects at all times. In this sense this algorithm can be called dual–primal. Step (2) corresponds to the primal problem (P3), while step (3) corresponds to the dual problem (D3).

(*d*) In step (0), the special choice of initial values $\{u\}$ and $\{Z\}$ gives a 'proximity' character to $\bar{G}$. u_i means, at the initial step, a kind of 'action radius'. Later (steps (2) and (3)) $\bar{G}$ is considerably modified, and a sequence of different matchings is thus obtained.

(*e*) Finally, it is interesting to make some comparisons between the dual–primal algorithm described above and the primal–dual one. In the first case, the algorithm yields different unperfect matchings during the course of the computation and the perfect minimum-weighted matching is obtained only in the last step. Then we start with a dual solution, and we obtain a primal solution.

In contrast to this procedure, the primal–dual algorithm starts with a given perfect matching, and progresses by improving this solution. This procedure seems to be analogous at first sight to the spirit of the relaxation techniques. But, in contrast to standard relaxation procedures, which fail to give ground states, the primal–dual algorithm yields a matching of minimum weight, i.e. ground states.

3.3. Practical implementation

The first algorithm we have implemented was the original one of Edmonds, because it needs much less memory storage than Lawler's one. The solving time on a 22×22 sample (at 15% of negative interactions) was about 20 s on an IBM 360–67.

A primal version of this algorithm, working on sample family, takes 150 s to solve 30 lattices 22×22 with $0{\cdot}10 \leqslant x \leqslant 0{\cdot}18$. In practice, it turns out that the likelihood of matching two plaquettes distant by more than five was null. We use this property to accelerate running and save storage by considering only the edges of K for which the weights are less than or equal to five.

4. Spin clustering

The occurrence of more than one chain between two frustrated plaquettes (see § 1) is

the cause of the multiplicity of solutions. In order to be more precise, let us consider in the case $J_{ij} = \pm 1$ two minimum-weighted perfect matchings M and M'. For each of these two matchings two precise configurations of spins exist.

Let $M \Delta M'$ be the symmetric difference between M and M'. It is clear that the edges in $M \Delta M'$ belong to alternating cycles of null weight relative to M and M'.

If we draw these cycles on the grid, we obtain a set of spin clusters surrounded by null cycles with the following property: by passing from M to M', each cluster is reversed in a coherent way. It follows immediately that all ground states in a frustrated network can be generated by reversing spin clusters delimited by null cycles.

This procedure yields all ground states, and leads to an estimation of the degeneracy of these states in the frustration problem (figure 5).

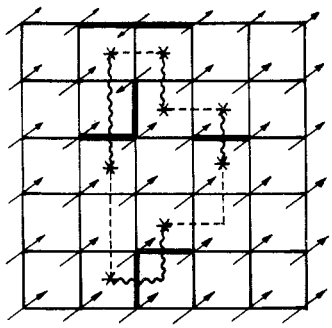

Figure 5. Alternating cycle obtained from two matchings, with spin configurations. The cluster of solidary spins is surrounded by the alternating cycle of asterisks and broken lines.

In this section we shall examine these spin packets in more detail, and we propose to describe a method yielding some information about the degeneracy of the ground states. This procedure can be used (see the following section) to give a description of the behaviour of spins in the ground states.

Let $K = (F, F \times F)$, the graph of frustrated plaquettes. We denote by A^* the subset of edges in K, inside a minimum-weighted perfect matching. Let K^* be the graph (K, A^*). If we draw K^* on the network G, replacing each edge by the shortest chain in G', we then see that each null cycle surrounds a spin packet. Moreover, *the maximal set of spins surrounded by connected components of K^* contains the union of all spins that can be contained in a spin packet.* It is possible to have more than one null cycle inside a connected component, but spins which are outside the set of connected components cannot be contained in a spin packet. In fact, there would otherwise exist a null cycle surrounding these spins, and then they would belong to a connected component.

It is easy to find an efficient procedure of polynomial complexity yielding the set A^*. We limit ourselves in this paper to the computation of a larger set A' containing A^*, using a simple algorithm described below. This algorithm is a subproduct of Edmonds' algorithm, and yields an efficient method for searching eventual fracture lines in the frustration problem (see § 5).

Let $\bar{G}^* = (F, E^*)$, the last equality graph given by Edmonds' algorithm. We denote by (U, Z) the last solution of the dual problem (D3).

Using the remark quoted in § 3 concerning the optimality criterion we can show that a perfect matching in G is of minimum weight if and only if it implies a perfect matching in $\bar{G}$ for which (C9) is satisfied. It also follows that A^* is none other than the set of edges

contained in alternating cycles relative to a given matching M. Each of these cycles has zero or one edge outside M, and only one end in each odd subset S for which $Z_S < 0$.

The set $A' \subset A^*$ has been obtained by deleting in $\bar{G}$ only the edges which are not contained in an alternating cycle. In this way, we add to alternating cycles some edges as shown in figure 6.

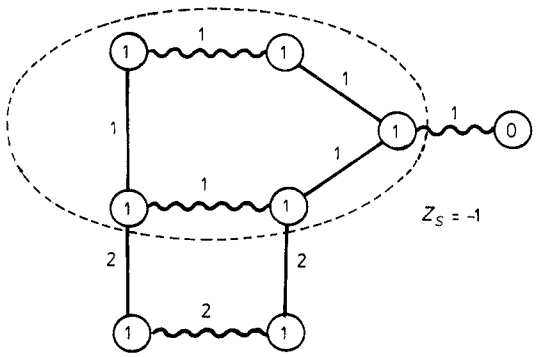

Figure 6. Construction of the set A^* from $\bar{G}$, with the blossom marked by a broken curve.

In order to obtain A', the following algorithm has been used for each vertex f in $\bar{G}^*$ which is an articulation point†.

(1) Let $C(f)$ be the connected component of $(F-\{f\}, E_f^*)$ containing the vertex connected to f by the matching M.

(2) Delete in $\bar{G}^*$ all edges which are of the form (f, f'), where f' is not a vertex of $C(f)$.

Here $E^*(f)$ is the set of edges in E not incident to f.

The computation of A' with this procedure yields two pieces of information. First, we can find the fraction of spins which is embedded in at least one spin packet. In fact, such spins are surrounded by the connected components. In the second place, A' gives the eventual fracture lines in the sample. In fact, the edges of such a line certainly belong to a 'percolating' connected component. It also follows that the search for this line is confined in this 'percolating' component (see § 6).

In the next section, we discuss the direct relationship between the fraction of spins in clusters and the probability for the existence of fracture lines. Finally, the search for all clusters, and also the degeneracy of the ground states, is a very difficult problem. At the present time, we are unable to count the number of solutions exactly. However, the above procedure gives without ambiguity all the spins remaining together in all ground states (solidarity).

5. Results and discussion

5.1. The energy E_0 of the ground state (per spin)

For each concentration of negative bonds, between $x = 0{\cdot}10$ and $0{\cdot}30$, the ground-state energy E_0 (per spin) is calculated for ten different samples generated independently (as

† A vertex of a graph is called a cut vertex or articulation if its removal results in an increase in the number of connected components.

described in the previous section). We have reproduced only the results for squares of size 22×22 spins in figure 7. The bars represent the standard-deviation error from the expectation for ten samples. This standard deviation gives a relative error of the order of 1·7%.

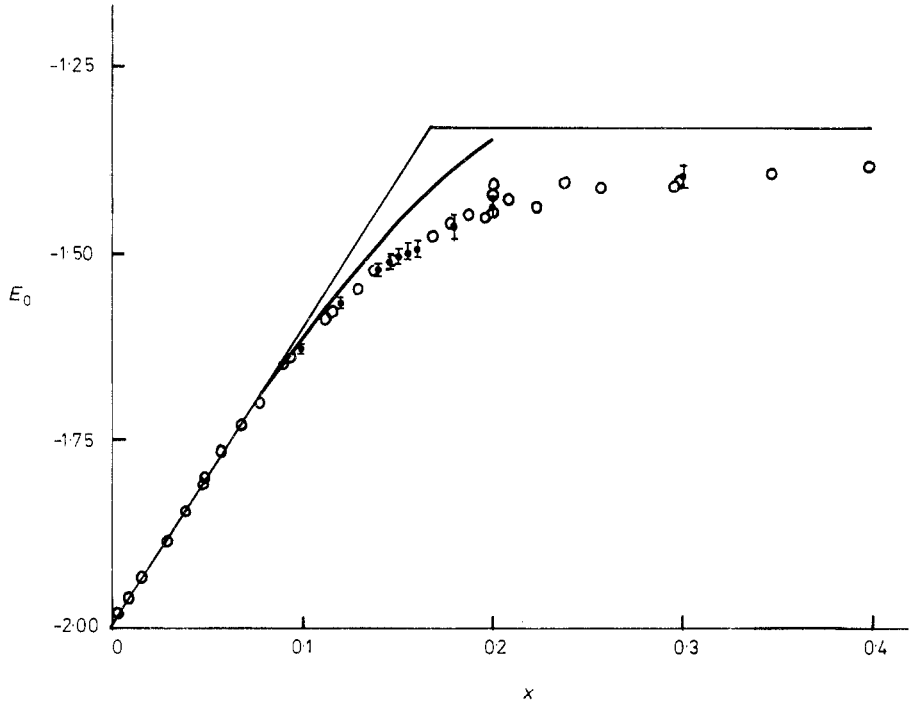

Figure 7. Ground-state energy (per spin) as a function of x. Ground-state energies as obtained by the matching method are represented by small full circles where the bars represent the standard deviation for ten different samples of size $N = 22 \times 22$. There is no perceptible anomaly on the curve around $x^* = 0{\cdot}145$. The open circles correspond to the results of Kirkpatrick (1977) using the Monte-Carlo method on a sample of size $N = 80 \times 80$. The curve corresponds to the series expansion $E_0(x) = -2(1-2x) - 16x^3 - 64x^4$ (Grinstein *et al* 1979).

The dispersion of the data is remarkably weak. This was not the case for the simulation by Monte-Carlo relaxation (improved by thermal shocks) on a bigger sample (80×80). These points have been reproduced for comparison in figure 7: the departure from the ground-state values must originate from the potential barrier which prevents the relaxation operating towards the ground states.

In figure 8, the same ground-state energy is plotted as a function of the concentration of frustrated plaquettes. But, as the sample was generated for a fixed value of x, a fluctuation in values of C_p appeared and it was necessary to present the results in the form of a histogram of width 0·01. An approximate linear variation of E_0 with C_p has been found which is compatible with the previous results of Vanimenus and Toulouse (1977).

In figure 9, E_0 has been plotted as a function of $1/N^2$, the inverse of the total number of spins in the samples, for various concentrations. The purpose of this plot is to verify

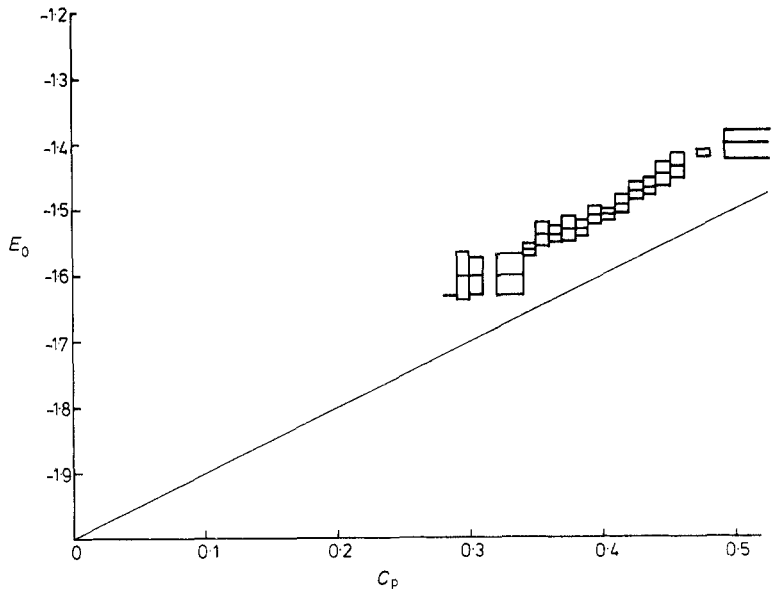

Figure 8. Ground-state energy (per spin) as a function of C_p. As the samples have been generated for fixed values of x, the concentration of frustrated plaquettes fluctuates to a large extent. All the samples are represented in a histogram of width 0·01. A linear regression gives $E_0(C_p) = -1{\cdot}9147 + 1{\cdot}0228C_p$, a variation compatible with the previous results of Vannimenus and Toulouse (1977). The line corresponds to the limit of large dilution of negative bonds: $E_0(C_p) = -2 + C_p$. $N = 22 \times 22$.

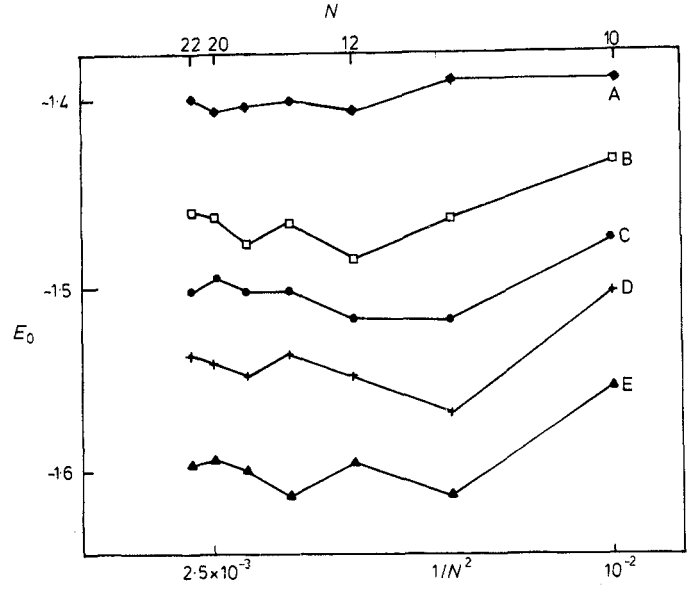

Figure 9. Size effect on the ground-state energy. Each point corresponds to the average of the ground-state energy on ten samples of variable size $N \times N$ but fixed density x. A, $x = 0{\cdot}30$; B, $x = 0{\cdot}18$; C, $x = 0{\cdot}16$; D, $x = 0{\cdot}14$; E, $x = 0{\cdot}12$.

that the size of samples, 22×22, which is selected for the exhibition of $E_0(x)$ in figure 7 is located in a range of values of $1/N^2$ where the fluctuations are rather small, in such a way that it reasonably represents the limit $N \to \infty$. In addition, this plot does not reveal any 'abrupt' change of regime of $E_0(1/N^2)$ which could have been attributed to a critical size regarding a characteristic coherence length.

For a dilute concentration, $x \sim 0$, it is possible to establish an expansion in x: $E_0(x) = -2 + 4x - 16x^2 - 64x^4$ (Grinstein *et al* 1979), which is represented by the full curve in figure 7. From this general behaviour of $E_0(x)$, no singularity of any order can be deduced from the data. The temporary conclusion is that the ground-state energy $E_0(x)$ is not a relevant function of the phase transition at the threshold of disappearance of ferromagnetism.

5.2. Probability of occurrence of a fracture line

The fracture line has been defined in § 4 as the longest alternating cycle running through the sample from one side to the opposite one. This fracture line permits the reversal of the spins in the left part of the sample when the right part is fixed, for instance: it is the sign of breaking the long-range order of the ferromagnetism phase. The study of this fracture line has been made by hand in the largest connected component ('percolating' component). For each concentration x, the line has been searched for in the ten samples and the probability of occurrence $P(x)$ is defined as the ratio of the number of fractured samples divided by ten. The results are exhibited in figure 10; $P(x)$ is expected from an infinite sample to be a step function defining unambiguously the critical concentration x^* for the disappearance of ferromagnetism. The finite-size simulation is obviously rounded, but the data show an abrupt variation near 0·145. This feature is magnified on the inset to figure 10, where the rate of variation of this probability (derivative dP/dx) with x is plotted. The critical value x^* is therefore in the range $0{\cdot}14 < x^* < 0{\cdot}15$, which corresponds to a new determination of x (the previous determinations were $0{\cdot}15 < x^* < 0{\cdot}20$ (Kirkpatrick 1977), $x^* \simeq 0{\cdot}15$ (Ono 1976) and $x^* \simeq 0{\cdot}10$ (Vanimenus *et al* 1979)). The present study therefore improves the accuracy of x^*. This value is discussed briefly in the following section.

5.3. Fraction of spins inside the connected components

The connected components refer to subsets of plaquettes which are matched together in any perfect matching of the graph. These components are surrounded by alternating cycles: they correspond to clusters of solidary spins, the reversed state of which has the same energy as the initial state. At low concentrations of negative bonds the largest connected component is considered as the reference state, which is assumed to be rigid, for instance, by fixing the spins on the outside perimeter of the sample. By increasing x, the largest component is still assumed to be the rigid reference state and is not taken into account in the fraction of spins in a cluster. This fraction, $F(x)$, gives a strong indication of the looseness of the spins in the ground states. In this context of Bray *et al* (1978), beyond the threshold each spin belongs to a finite cluster in such a way that $F(x) = 1$ for $x > x^*$.

The present definition of the cluster of spins must be distinguished from the concept of the packets of solidary spins as introduced by Vanimenus *et al* (1979). The packet of solidary spins is defined as a group of spins that always keep the same relative orientation. These packets have a correlation equal to 1 at $T = 0$, and they provide a

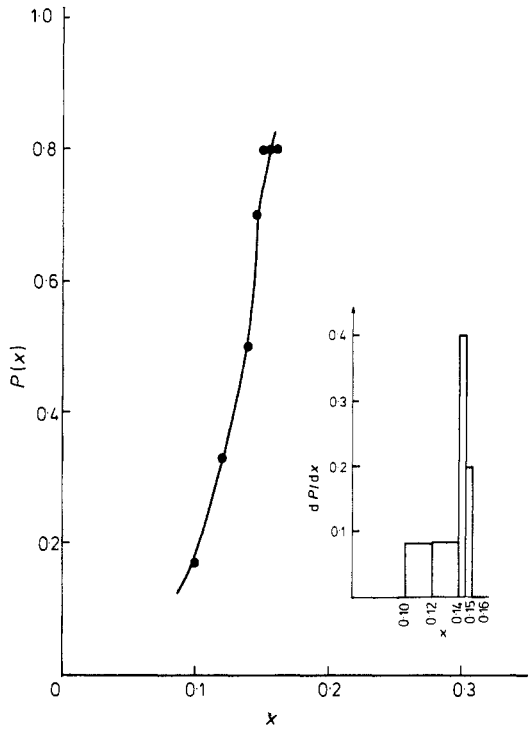

Figure 10. Probability of occurrence of a fracture line. The fracture line—alternating cycle running from one side of the sample to the opposite—is searched for on each set of ten samples corresponding to a fixed x. $P(x)$ is defined as the ratio of fractured samples divided by ten. The continuous line is only a guide to the eye. The insert represents the rate of increase dP/dx which exhibits a peak at $x^* = 0{\cdot}145$, the threshold for the disappearance of ferromagnetism. $N = 22 \times 22$.

measurement of the rigidity in the ground states. Therefore, the fraction of spins which belong to at least one packet is a decreasing function of x from 1 at $x = 0$, in contrast to $F(x)$. There the solidarity of spins is a measurement of the rigidity, while here it is related to the looseness of the ground states.

From the algorithm it is possible to determine the total number of spins in the connected component. This number, averaged over the ten samples and divided by the total number of spins, is by definition the fraction of spins inside the connected components $F(x)$. $F(x)$ against x has been plotted in figure 11 for the biggest sample, $N = 22 \times 22$.

The data exhibit a steep variation near $x = 0{\cdot}14$–$0{\cdot}15$. The derivative of this curve is shown in the inset, and reveals a sharp peak at $x^* = 0{\cdot}145$ which is the same value as determined by the previous curve, figure 10. Both properties—the probability of occurrence of fracture lines and the fraction of spins—must be considered as relevant variables for the phase transition between the ferromagnetic and the 'spin glass' phases (it can be noticed, however, that it is easier to obtain $F(x)$ than $P(x)$).

5.4. Magnetisation

The magnetisation $M(x)$ is defined as the difference between the number of spins in the

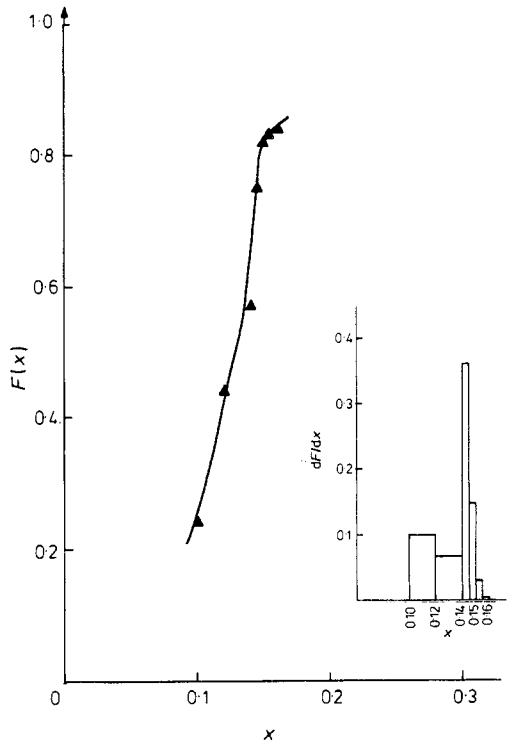

Figure 11. Fraction of spins in the connected components. The connected components of the graph describe the clusters of solidary spins which can be reversed without cost in energy. The more spins there are in these clusters, the more loose are the ground states. For each x, the average number of spins belonging to these components is counted (over ten samples), except those of the largest component, which are taken as rigid, and the fraction $F(x)$ is the ratio of this average number over the total number of spins. The full curve is only a guide to the eye. The insert represents the rate of increase dF/dx which is strongly peaked at $x^* = 0{\cdot}145$, a value considered consequently as the threshold for the disappearance of ferromagnetism. $N = 22 \times 22$.

two states. For a fixed reference direction of the spins (± or up to down) $M(x)$ can be positive or negative. However, only the absolute value of $M(x)$ can be studied when a comparison or an average between different samples is performed and $\langle|M(x)|\rangle$ is obtained. The expectation value of the absolute value of the magnetisation for ten samples of fixed concentration and size is plotted in figure 12. It must be realised that this averaging procedure is only partial in the sense that, for a fixed sample, the magnetisation must also be averaged for all the ground states. But this cannot be done since the algorithm cannot generate the total number of ground states. The sample averaging is probably a good approximation of the correct expectation value of the magnetisation for small x ($\lesssim 0{\cdot}1$) in the ferromagnetism phase, where both the degeneracy and the fluctuation of the magnetisation are weak (few small packets). But for larger values of x, particularly near and above the threshold, this $\langle|M(x)|\rangle$ is largely overestimated. In this case, the ground states are numerous and their magnetisation fluctuates not only in intensity but also in sign for fixed spin reference, leading to an expectation value $\langle M(x)\rangle = 0$ above the threshold. For this reason, we believe that the

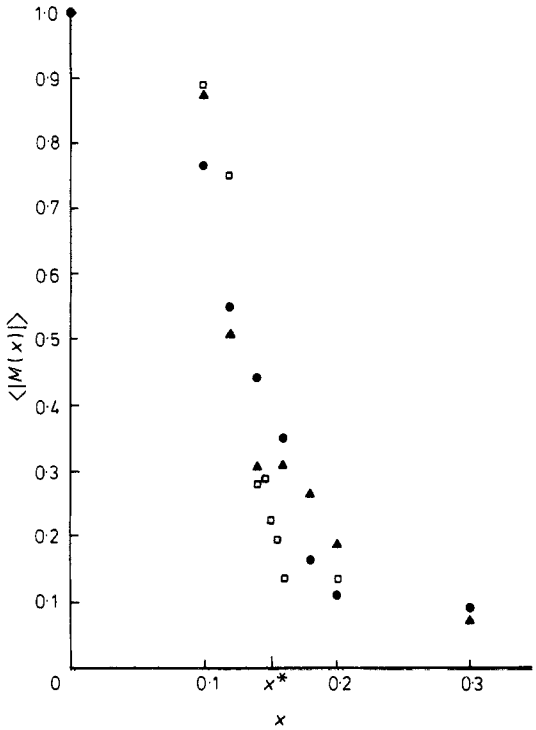

Figure 12. Magnetisation. The absolute value of the magnetisation (per spin) is averaged over ten samples of fixed x to give $\langle|M(x)|\rangle$. For each size the results are represented by triangles, full circles and squares for $N = 18\times18$, 20×20 and 22×22 respectively. Due to the fact that only the absolute value is meaningful for comparison between different samples, $\langle|M(x)|\rangle$ is overestimated above x^* where $\langle|M(x)|\rangle$ must be well in the limit $N\to\infty$.

data are meaningful below x^* but largely overestimated above x^*. Despite this difficulty, the curves shown in figure 12 exhibit an abrupt decrease of $\langle M(x)\rangle$ near x^*. This decrease is amplified for the largest sample, $N = 22\times22$, as expected.

6. Conclusion

Edmonds' algorithm—perfect matching of minimum weight—has been applied to the frustration model in order to obtain the ground states of a spin glass. As compared with other numerical methods of relaxation, the advantage of this algorithm is to generate the true ground states for samples of finite size. The main result of this study is the determination of the threshold for the disappearance of ferromagnetism for the critical concentration of negative bonds $x^* = 0{\cdot}145\pm0{\cdot}01$. We have no physical explanation of this value, but it might be compared with the critical value of the annealed model of frustration. By the annealed model, we mean a model where all pair interactions J_{ij} between neighbouring spins are considered as internal variables, like the spin variables, with the constraint of a fixed value for the density of negative bonds. It has been shown (Thorpe and Beeman 1976) that this annealed $\pm J$ square lattice model has an exact solution which can be mapped into the Onsager solution of the regular spin lattice, but

with an effective interaction depending on the distribution of negative interactions as well as on the temperature. An implicit equation for this effective interaction can be written at $T = 0$; the solution of this is, in the $\pm J$ model, simply $x^* = \frac{1}{2} - \frac{1}{4}\sqrt{2} \simeq 0{\cdot}1464$. It is remarkable that this value falls in the range of the present determination and raises the question of the possible coincidence of the threshold between the annealed and quenched models of frustration. We have not found any symmetry argument, from duality for instance, on the square lattice which could justify this identity of thresholds. In this context it can be recalled that for dilute Ising systems in a square lattice—the model where a density x of bonds is such that $J = 0$—the threshold of the annealed model is $x_c = 0{\cdot}5$, the same as the one for the bond percolation problem considered as the quenched model of the dilute Ising problem. However, for a triangular lattice of dilute Ising systems the threshold is 0·3522 for the annealed model instead of 0·3473 for the quenched one (or bond percolation problem). The possibility of a threshold for the quenched $\pm J$ model slightly below but different from the annealed model one (0·1464) cannot be eliminated.

Another important result of this study is the occurrence of fracture lines, a signature of the disappearance of the long-range order of ferromagnetism. This occurrence above x^* is accompanied by the clustering of solidary spins in packets in the sample, which reflects the looseness of ground states. This phenomenon could be very general in this type of disordered systems (Toulouse 1979)—three dimensions, Heisenberg spins, $p(J)$ continuous—and can be understood as a characteristic size for screening the strong but competing interaction between spins. If these packets do not overlap in such a way that they realise a partition of the spin systems, the problem of phase transition would have been solved in a negative way: no phase transition can occur for independent and finite packets of solidary spins (in thermodynamic equilibrium). But the overlapping of packets prohibits such a simple conclusion!

To our knowledge, this is the first time that this algorithm has been applied to a physical problem. One wonders if other models can be formulated in terms of matching problems. The dimer problem—pairing two-neighbour sites on a lattice—is obviously a perfect matching problem, but the interest here is the enumeration of states which cannot be treated directly by this algorithm which generates a perfect matching of minimum weight.

Appendix. Two illustrative examples

The purpose of this appendix is to show how the algorithm described in § 4 works. For this, we shall consider two cases of minimum-weighted perfect matching. The first is related to bipartite graphs, the second to non-bipartite graphs. In order to understand this procedure, it is profitable for the reader to consider these examples as exercises.

A.1. Bipartite graphs

Let us consider the bipartite graph shown in figure 13(a), where the vertices are numbered 1, 2, 3, . . . , 8, and different edge weights are indicated.

A.1.1. Initialisation of the values u_i and construction of $\bar{G}$ (steps (0) and (1)). Applying steps (0) and (1), we obtain the graph $\bar{G}$ shown in figure 13(b). It is interesting here to

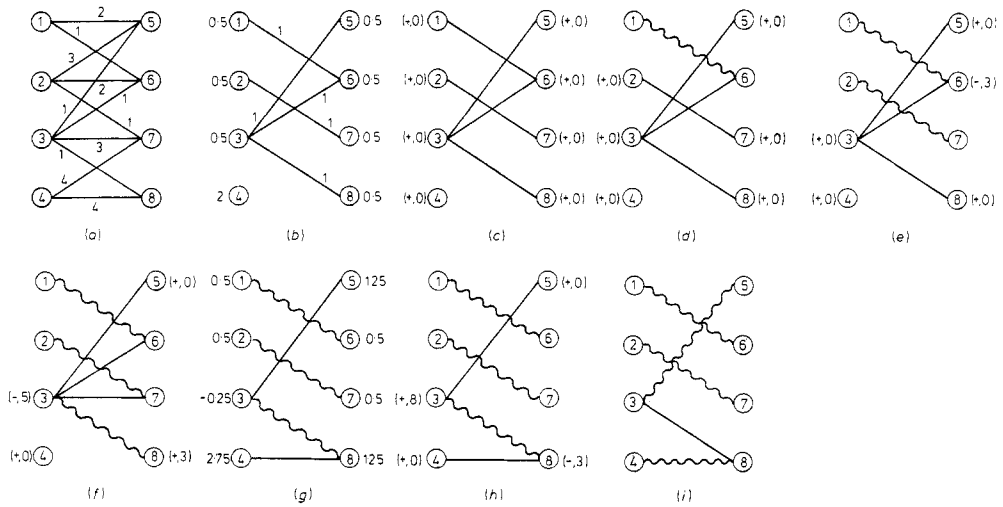

Figure 13. Bipartite graphs.

note the absence of odd subsets, S. Therefore $Z_S \equiv 0$ at all times during the course of the computation.

A.1.2. Research of a maximum cardinality matching in $\bar{G}$ (step (2)). For this, we apply one of standard algorithms, given such matching (Lawler 1976). For example, we can use the following procedure (labelling algorithm).

(0) Let M be any matching in $\bar{G}$, possibly the empty matching. No vertices are labelled.

(1) Give the label (+, 0) to each exposed (unsaturated) vertex.

(2) If there are no unscanned, labelled vertices, stop. Otherwise, if there exists a vertex i labelled +, and unscanned, go to (3) for scanning. Otherwise, find a vertex j, labelled −, unscanned, and go to (4) for scanning.

(3) For each vertex j adjacent to i, with an edge outside M, then give the label $(-, i)$ to j if j is unlabelled or, if j is labelled +, an augmenting alternating chain is detected and go to (5). Go to (5).

(4) Give the label $(+, j)$ to the vertex i, matched to j by M. Go to (2).

(5) Using the labels, find the augmenting alternating chain. Increase the cardinality of the matching. Remove all labels from vertices. Go to (1).

In the case considered here, initially, M is empty. In the figures, wavy lines represent edges in the matching M, and straight lines those which are not in M.

Following the prescriptions above, we encounter successively the following steps.

(*a*) By scanning the label + at vertex 1 we detect the augmenting chain $\{(1, 6)\}$ (figure 13(*c*)).

(*b*) By scanning the label + at vertex 2 we detect the augmenting chain $\{(2, 7)\}$ (figure 13(*d*)).

(*c*) By scanning the label + at vertex 3 we give the label (−, 3) to vertex 6, and we also detect the augmenting chain $\{(3, 8)\}$ (figure 13(*e*)).

(*d*) Scanning the label + at vertex 4 does not generate any new label.

(*e*) By scanning the label + at vertex 5 we give the label (−, 5) to the vertex 3.

(f) By scanning the label − at vertex 3 we give the label (+, 3) to the vertex 8.

(g) Scanning the label + at the vertex 8 does not generate any new label.

At this stage, we obtain the matching $M = \{(1, 6), (2, 7), (3, 8)\}$, which is of maximum cardinality in $\bar{G}$ (figure 13(f)).

The following step is step 3 in Edmonds' algorithm, where we change the values of u_i'. For this, we can use the last labels of the vertices in $\bar{G}$, and we can make $u_i = u_i + \epsilon$ for every vertex i labelled +, and $u_i = u_i - \epsilon$ for every vertex labelled −, with $\epsilon = \min(\epsilon_1, \epsilon_2)$. Here ϵ_1 and ϵ_2 are defined as follows. ϵ_1 represents the minimum value of $[W_{ij} - (u_i + u_j)]$, where (ij) represents an edge such that i is labelled +, j is unlabelled and $u_i + u_j < W_{ij}$ is satisfied. ϵ_2 represents half of the minimum value of $[W_{ij} - (u_i + u_j)]$, where (ij) represents an edge such that i and j are labelled + and $u_i + u_j < W_{ij}$ is satisfied. In our example, $\epsilon_1 = \min(1, 2) = 1$, $\epsilon_2 = \min(0{\cdot}75) = 0{\cdot}75$. Then $\epsilon = 0{\cdot}75$, and we start with the new equality graph $\bar{G}$ as shown in figure 13(c). This new graph contains the edge (4, 8), but does not contain (3, 6).

Repeating the procedure described above, we obtain a new matching of maximum cardinality in $\bar{G}$ (figure 13(h)).

Now we detect an augmenting alternating chain $\{(4, 8), (8, 3), (3, 5)\}$ and also change the new matching (figure 13(i)).

The next matching M obtained is also perfect in $\bar{G}$. From § 4, M is then a perfect matching of minimum weight in G. Thus the procedure stops at this step, with the solution also obtained. The weight of M is 7.

A.2. Non-bipartite graph

Let us consider the non-bipartite graph G, shown in figure 14(a), with a weight assigned to each edge. After the initialisation of the values u_i, we obtain the graph $\bar{G}$ as shown in figure 14(b). $M = \{(1, 2), (3, 6)\}$, which represents a matching of maximum cardinality in $\bar{G}$, can be chosen as the starting point (figure 14(c)).

When we scan the label − of the vertex 1, we remark that the vertex 4 is labelled by (−, 3). However, this vertex can be labelled (+, 2). This conflict is the signature of the presence of an odd cycle. The set of vertices belonging to this cycle satisfies the equality

$$\sum_{e \in E(S)} x_e = \tfrac{1}{2}(|S| - 1).$$

Such a cycle is called 'blossom', and is rooted at the unsaturated vertex by an element of $E(S)$.

In our example, we have two blossoms: $B_1 = (1, 2, 3)$ rooted at $b_1 = 3$ and $B_2 = (4, 5, 6)$ rooted at $b_2 = 4$ (figure 14(d)). For each vertex in a blossom, other than the root, we continue the labelling operation. Moreover, in the following step, each blossom is to be considered as only one vertex, labelled with its root label.

The matching obtained with this prescription has the maximum cardinality. In the next step, we modify the dual solution (u, Z). For this we take

$$\epsilon = W_{b_1 b_2} - (u_{b_1} + u_{b_2}).$$

Also, for each vertex i in the blossom B_1, we make $u_i = u_i + \epsilon$, because b_1 is labelled with the label +. Finally, we change Z such that $Z_{B_1} = -2\epsilon$ from the condition (C8).

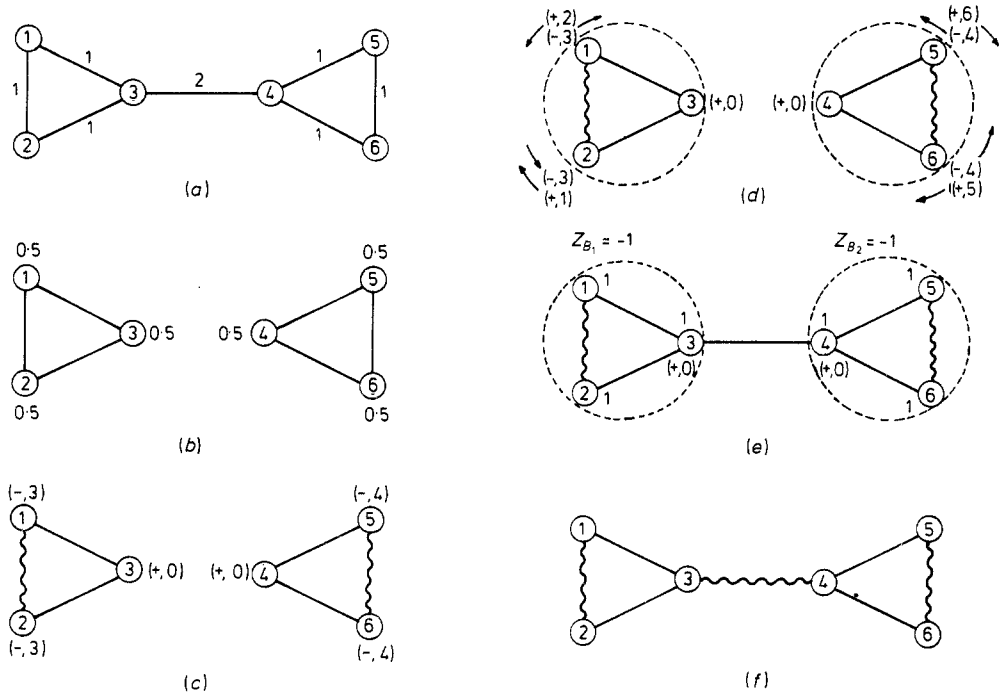

Figure 14. Non-bipartite graphs.

In a similar way, for each vertex in the blossom B_2, we make $u_i = u_i + \epsilon$, and Z is changed such that $Z_{B_2} = -2\epsilon$. Therefore, the graph $\bar{G}$ becomes as shown in figure 14(e).

In $\bar{G}$ an augmenting chain $\{(3, 4)\}$ can be detected. Finally, the next matching thus obtained is perfect, and of minimum weight, as we can see directly (figure 14(f)).

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